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Finding the location of the axes, the vertices and the foci of a parabola, an ellipse and a hyperbola using a straightedge and a compass

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1. Abstract

We present geometrical constructions using a straightedge and a compass in order to find the location of the axes, the vertices and the foci of a parabola, an ellipse and a hyperbola from their plots. The constructions are based on familiarity with theorems and special properties characterizing these loci, which therefore can be used for implementing and applying knowledge acquired during the studies of analytical geometry.

2. Introduction

Geometrical constructions comprise a fascinating and surprising world which has been the focus of attention of mathematicians since ancient times. As time went on, occasionally new methods were found to carry out additional constructions which were based on the discovery of geometrical properties [1-3] and original ideas based on them, which made such constructions possible. Among the surprising findings were the possibilities of construction in which the drawing tools were restricted, such as construction using a straightedge only or construction using a compass only [4-6]. Even today, when there is a sophisticated computer technology, the geometrical constructions in the traditional drawing tools, continue to employ mathematicians [7-9].
Students can usually carry out basic constructions such as: copying a segment, copying an angle, bisecting a segment, bisecting an angle, constructing a perpendicular to a segment at a given point, dropping a perpendicular to a segment from a given point, constructing a line that is parallel to a straight line, constructing triangles from data etc. Aside for the construction of segment copying, the rest of these basic constructions are based on acquaintance with theorems in geometry, each of which

characterizes a geometrical property. For this reason, the ability to make geometrical constructions is in fact the ability to implement and apply theorems in geometry. Therefore, one can consider geometrical constructions to be a field in which knowledge and theorems in geometry can be practiced and the level of their knowledge can be checked.

By dealing with geometrical constructions the student can find several methods of construction for the same task or find unconventional methods of construction and thereby shed light on the fascinating beauty of mathematics.

The subject of the parabola, the ellipse and the hyperbola is studied at the ages 16-17, as part of the studies of analytic geometry. At the first stage the definition is given for each of the loci, and subsequently the mathematical equation of each figure is developed. After basic practice, the students are directed to discover hidden geometrical properties, some of which are common to the two figures or more, such as: “the middles of all the chords in the hyperbola, which are parallel to a given straight line are located on a second straight line”, or “the sum of the squares of the inverses of the lengths of the chords that are perpendicular to each other in an ellipse, and which pass through its center – is a constant value”.

These loci have many special geometrical properties. Some of these properties can be applied to the construction of geometrical constructions, thus realizing knowledge and integrating different fields of mathematics.

As part of a course of geometrical constructions taught to pre-service teachers of mathematics late in their training, an activity took place in which constructions were made with the curves of the following loci given: a circle, a parabola, an ellipse and a hyperbola. These were taught as part of a course in analytical geometry, and one had to find the location of special points such as foci and vertices by construction, together with their parameters.

2.1. Using a dynamic geometry environment (DGE)

The recent introduction of dynamic geometry software (DGS), for example GeoGebra, into classrooms creates a challenge to the praxis study and teaching of Euclidean geometry. Students/learners can experiment through different dragging modalities on geometrical objects that they construct, and consequently infer properties, generalities, and conjectures about the geometrical artifacts.

The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence, the students become convinced that their conjecture will always be true [10]. Nevertheless, because of the inductive nature of the DGE, we entitle this process ‘semi proof’. Hence, following the employment of DGE, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological concern [11]. Students must be aware that they still need to prove rather than rely on the virtual experiment. The DGE software serves as an intermediary tool used to bridge the gap between the geometrical model and the geometrical constructions

As an introduction to the unique geometrical constructions that will be presented in the paper, we first present a simple construction whereby one must find the center of a circle using drawing tools (straightedge and compass).

Using a straightedge, we mark two chords, AB and CD on the circle, and construct the mid-perpendicular of each (see Figure 1).

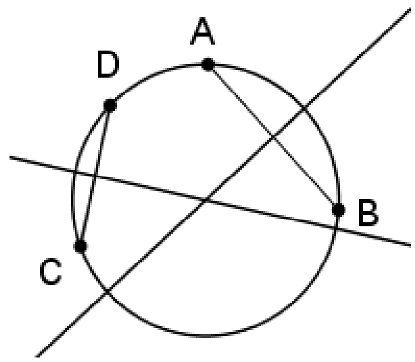

Fig. 1: Finding the center of a circle using a straightedge and compass

Based on the theorem "the center of a circle lies on the mid-perpendicular to a chord", the point of intersection of the two mid-perpendiculars is the center of the circle.

An interesting challenge is to find the location of the axes and the foci of a parabola, an ellipse and a hyperbola, since in this case the construction is more complicated and requires acquaintance and the ability to apply geometrical properties at each stage of the construction in which use is made of a particular geometrical property that characterizes the locus [1-3]. Attached are links to GeoGebra applets which illustrate the constructions and allow the reader to carry out the dynamic investigation of the different properties.

3.2. Task A:

A1: Finding the symmetry axis and the vertex of a parabola

Figure 2a shows the curve of a parabola without marking the symmetry axis and without marking its vertex.

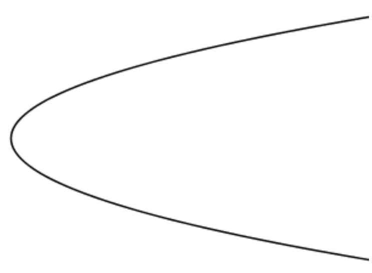

Figure 2a: a parabola

Figure 2b shows the parabola after the following construction.

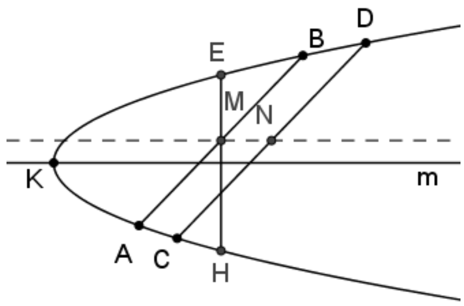

Figure 2b: Finding the symmetry axis
and the vertex of a parabola

<u>Description of the construction</u>

We construct in the parabola two arbitrary parallel chords AB and CD. The middles of these chords are M and N, respectively. It is well known that the middles of parallel chords in a parabola lie on the straight line that is parallel to the axis of the parabola [1]. In other words, the straight line passing through the points M and N is parallel to the axis of the parabola. At the point M we construct a perpendicular to the straight line MN, which intersects the parabola at the points E and H. We construct the straight line m, which is the mid-perpendicular to the segment EH. The straight line m is the symmetry axis of the parabola at the point of its intersection with the parabola, K, is the vertex of the parabola.

Link: 1: **A straight line that connects the middles of parallel chords in a parabola**

https://www.geogebra.org/m/csdNrzft

Applet 1 that can be reached by link 1, illustrates the property that the straight line that passes through the midpoint of a pair of parallel chords is parallel to the axis of the parabola. By dragging the point A or the point B on the curve of the parabola, one can change the inclination angle of the parallel chords. By dragging the point C one can change the distance between the parallel chords. One can change the value of the parameter of the parabola p using a slide bar.

A2: Finding the location of the focus of a parabola

<u>Method A</u>: Finding the location of the focus using a straight line: The straight line is passed through the vertex K, whose equation is $y = x$. This straight line intersects the parabola $y^2 = 2px$ at $A(2p, 2p)$, as described in Figure 3.

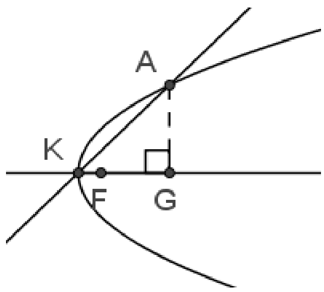

Figure 3: finding the focus of a parabola

From the point A we drop the perpendicular AG to the axis of the parabola. The length of the segment is $KG = 2p$. The segment KG is divided into four equal segments with a length of $KF = \frac{p}{2}$, the point F is the location of the focus.

Method B: Finding the location of the focus using a tangent and a normal.

A well-known property is that the tangent to a parabola at the point E on the parabola intersects the axis of the parabola at the point L, such that KG = KL, as described in Figure 4.

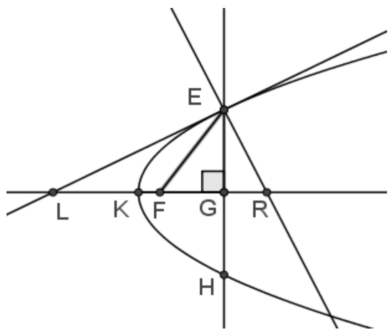

Figure 4: finding the focus of the parabola using a tangent and a normal

From the point K we draw a circle with a radius KG, obtaining the point L. We draw the tangent EL. At the point E we construct the normal to the parabola, which intersects the axis of the parabola at the point R. The segment GR is the projection of the normal on the symmetry axis, and it is the length of the parabola's parameter *p*. From the point K we draw an arc with a radius of *r*, $r = \frac{GR}{2} = \frac{p}{2}$, which intersects the axis of symmetry at the point F, which is the focus of the parabola.

Note: Since the triangle ΔEFL is an isosceles triangle, it is enough to construct the mid-perpendicular to the segment LE, which intersects the axis of symmetry at the focal point.

Link 2: Equality of segments on the axis of a parabola.

https://www.geogebra.org/m/MUdTrapS

Applet 2, which can be reached by Link 2, illustrates the conservation property of the segment lengths LK = KG, as the point E moves along the parabola. Using a slide bar, the parameter of the parabola can be changed.

Link 3: Constant length of the projection of the normal of a parabola

https://www.geogebra.org/m/E9VFPCkQ

Applet 3, which can be reached by Link 3, illustrates the property according to which the length of the normal's projection on the axis of the parabola is a fixed magnitude that equals the parameter of the parabola when the point L is dragged along the axis of the parabola.

Method C: Finding the location of the focus using the mid-perpendicular to a chord.

We draw an arbitrary chord AB, and construct the mid-perpendicular to that chord at the point M on the chord, as shown in Figure 5. The mid-perpendicular intersects the axis of the parabola at the point R.

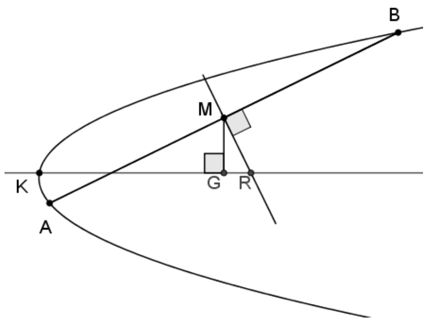

Figure 5: finding the focus of the parabola using the mid-perpendicular to a chord

From the point M we drop the perpendicular MG to the axis of the parabola.

From a known property, the length of the segment GR is p - the parameter of the parabola. To find the focus, draw an arc with a radius of $\frac{GR}{2}$ from the point K.

Link 4: Constant length of the projection of the mid-perpendicular on a chord in a parabola

https://www.geogebra.org/m/Y7RREbJU

Applet 4, which can be reached by Link 4, illustrates the property according to which the projection GR of the segment MR of the axis of the parabola is a fixed magnitude when the point B is dragged along the parabola, and provided that the chord AB is not perpendicular to the axis of the parabola. A slide bar can be used to change the parameter of the parabola, and at each stage the values of *p* and GR appear on screen.

<u>Other methods</u> (without descriptions of the construction and without figures)

1. Two chords that are perpendicular to each other are drawn in the parabola. Let M and N be the midpoints of the chords, then $y_M \cdot y_N = -p^2 \;\Rightarrow\; p = \sqrt{-y_M \cdot y_N}$, and the construction is carried out based on the values of y_M and y_N, and on their constructed geometrical mean.

2. Let A be some point on the parabola. The perpendicular from A to the *y* axis intersects the *y* axis at the point B. The perpendicular from the point B to the straight line AO (O is the origin) intersects the *x* axis at the point C, the coordinates are $C(2p,0)$.

3. Let A be some point of the parabola. A known property is that the tangent to the parabola at the point A intersects the *y* axis at the point B, when $\angle ABF = 90^{\circ}$. We construct the perpendicular to the segment AB through the point B. this perpendicular intersects the axis of symmetry at the focal point.

The variety of different ways that construction is carried out indicates the richness and beauty of mathematics [12].

3.3 Task B: Finding the axes of symmetry and the values of the parameters of an ellipse

Figure 6 shows an ellipse on which the symmetry axes are not marked.

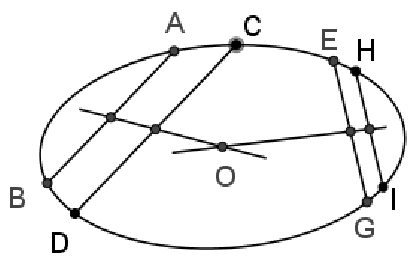

Fig. 6: Finding the center of an ellipse

Description of the construction

In the ellipse we construct a pair of parallel chords (AB and CD), together with their midpoints. We construct another pair of parallel chords (EG and HI), together with their mid-points. These chords are not parallel to the first pair.

Based on a known property of ellipses: "the middles of parallel chords are located on a straight line that passes through the center of the ellipse" we draw the two straight lines that connect the middles of the chords that intersect at the point O, which is the center of the ellipse (see Fig. 6). From the point O – the center of the ellipse, we draw a circle with radius, which intersects the ellipse at four points: A, B, C, D, as described in Figure 7. These points (due to the symmetry of the ellipse w.r.t. its axes) are the vertices of a rectangle whose sides are parallel to the axes of the ellipse.

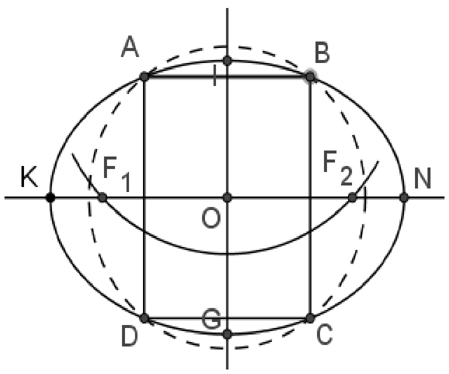

Fig. 7: Finding the foci of an ellipse

We connect the middles of the opposite sides of the rectangle, thus obtaining the axes of symmetry of the ellipse. These axes intersect the ellipse at the points K, I, N, G, which are the end points of the axes of the ellipse. Therefore, $a = ON$ is the parameter of the large axis of the ellipse, $b = OI$ is the parameter of the small axis of the ellipse.

From the definition of the ellipse we have $r_1 + r_2 = 2a$, where r_1 and r_2 are the radiuses issuing from the foci. For the point I there holds $r_1 = r_2 = a$. Therefore from this point we draw an arc with a radius of *a*, which intersects the large axis of the ellipse at two points, which are its foci (F_1 and F_2 in Fig. 7).

Link 5: Straight lines that connect the middles of parallel chords in an ellipse.

https://www.geogebra.org/m/y4NjzNuB

Applet 5, which can be reached by Link 5, illustrates the property according to which the middles of parallel chords in an ellipse are located on a single straight line that passes through the center of the ellipse. By dragging the point A or the point E on the ellipse one can change the angle of the chords. By dragging the points C or H one can change the distance between the parallel chords.

3.4 Task C:

C1: Finding the length of the real axis of a hyperbola – the value of the parameter *a*

In this case as well we have a hyperbola on which the symmetry axes are not marked. The center of the hyperbola (the point at which the axes meet) is found and the axes are drawn using the same method and based on the same properties which were used to find the axes of the ellipse (Task B). In finding the other parameters (*b* and *c*) there is a difficulty that did not occur in the case of an ellipse.

The parameter *a* is obtained from the point of intersection of the hyperbola with the real axis (the *x* axis), as described in Figure 8.

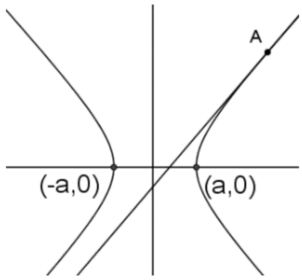

Figure 8: finding the real axis of a hyperbola

The parameter *b* cannot be obtained because the hyperbola does not intersect the imaginary axis (the *y* axis).

To find the parameters *b* and *c*, one must construct a tangent to the hyperbola.

<u>Description of the construction of the tangent to a hyperbola at a given point on it</u>

Let the point $A(x_A, y_A)$ be a point on the hyperbola at which we wish to construct a tangent to the hyperbola (see Figure 8).

The equation of the tangent is $\frac{x_A \cdot x}{a^2} - \frac{y_A \cdot y}{b^2} = 1$ and it intersects the *x* axis at the point $B(x_B, 0)$. From the equation of the tangent we obtain $x_B \cdot x_A = a^2 \Rightarrow a = \sqrt{x_B \cdot x_A}$.

Construction of the segment x_B : we copy the segment x_A and mark its ends by K, L (see Figure 9). At the point L we construct a perpendicular whose length is *a*. We obtain a right-angled triangle ΔKLM . From the point M we draw a perpendicular to the straight line KM that intersects the continuation of KL at the point N. the length of the segment LN is x_B, and since the triangle ΔKLM is right-angled, *a* is the geometrical mean of x_B, x_A.

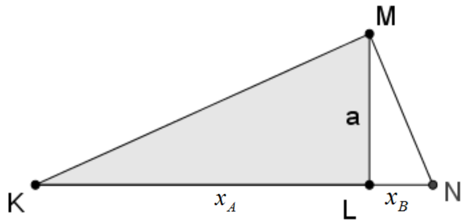

Figure 9: finding the length of the real axis of the hyperbola

The tangent can be constructed from the lengths x_A and x_B.

C2: Finding the values of the parameters *b* and *c* of a hyperbola using a tangent and a normal

We select an arbitrary point A on the hyperbola, at which we construct a tangent and a normal that intersect the real axis at the points Q and R, respectively, as described in Figure 10.

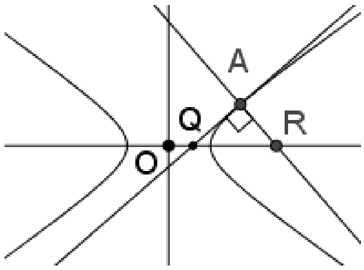

Figure 10: finding the values of the parameters *b* and *c* of the hyperbola

From a known property we have $OQ \cdot OR = c^2$, where O is the point of intersection of the axes. Therefore $c = \sqrt{OQ \cdot OR}$.

From the lengths of the segments OQ and OR, we construct their geometrical mean, which is the parameter *c*.

We now construct a right-angled triangle, one of whose legs is *a*, and whose hypotenuse is *c*, and obtain the other leg, *b*, which is the parameter of the imaginary axis.

Link 6: Constant product of segments on the axis of the hyperbola

https://www.geogebra.org/m/jB24sajs

Applet 6, which can be reached by Link 6, illustrates the property according to which the product of the distances of the points of intersection of the tangent and the normal (with the real axis) from the origin is a fixed magnitude that equals the square of the focal distance (c^2), where the point of the tangency A moves of the hyperbola. When the point Q is dragged the value of the product of distances $OQ \cdot OR = c^2$ remains constant.

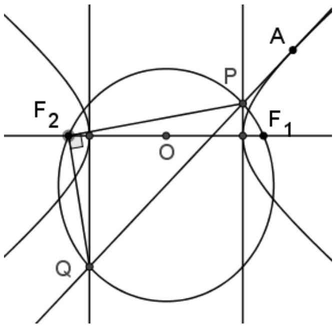

Figure 11: finding the values of the parameters *b* and *c* of a hyperbola using three tangents.

C3: Finding the values of the parameters *b* and *c* of a hyperbola using three tangents

We construct a tangent at a point A on the hyperbola, as well as two tangents at the vertices of the hyperbola as described in Figure 11. The tangent at the point A intersects the tangents that pass through the vertices at the points P and Q. Based on a known property, the segment PQ is observed from the foci of the hyperbola at a right angle.

On the segment PQ as a diameter we construct a circle that intersects the real axis at the points F_1 and F_2, which are the foci over the hyperbola. We now construct a right-angled triangle one of whose legs is *a*, and whose hypotenuse is *c* (the length of the segment $F_2O = c$). The other leg is the parameter *b*.

Link 7: View from the foci of the hyperbola cut off a tangent to the hyperbola.

https://www.geogebra.org/m/PsJ3bsZu

Applet 7, which can be reached by Link 7, illustrates the property according to which the segment cut off the tangent by two tangents to the hyperbola at its vertices can be observed at 90° from the foci of the hyperbola. When the point A is dragged on the hyperbola, the angle $\angle PF_2Q$ remains constant at 90°.

C4: Finding the values of the parameters *b* and *c* of a hyperbola without tangents

After finding the value of *a*, we find the *y* coordinate of a point A on the hyperbola whose *x* value is 2*a*, as shown in Figure 12.

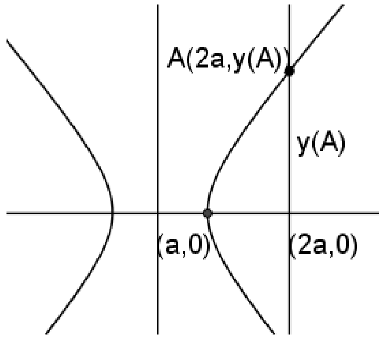

Figure 12: finding the parameters *b* and *c* of a hyperbola without tangents

When the coordinates of the point A are substituted in the equation of the hyperbola, we obtain:

$$\left(\frac{2a}{a}\right)^2 - \left(\frac{y_A}{b}\right)^2 = 1 \Rightarrow \frac{{y_A}^2}{b^2} = 3 \Rightarrow \frac{y_A}{b} = \sqrt{3}$$

To find the value of *b*, will make use of the known geometrical property, that in the right triangle whose angles are 30°, 60°, 90°, the ratio of the legs is $\sqrt{3}$.

We copy the segment y_A, whose ends are the points B and C, as shown in Figure 13. From the vertex C we draw a perpendicular and at the vertex B will construct an angle of 30°. The ray of the angle from the point B intersects the perpendicular drawn at the point C at the point A. The length of the leg AC is the magnitude of the parameter *b*. The length of the parameter *c* (the location of the foci) is found using the same method as described in Section C2.

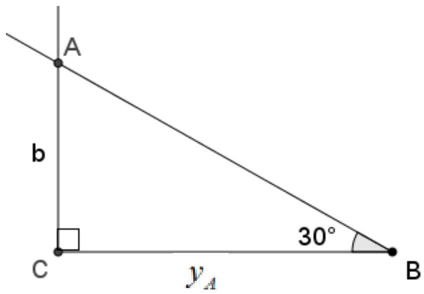

Figure 13: finding the values of the parameter *b* of the hyperbola

3. Methodical note

The constructions are made possible by employing the special properties of the loci. Even we, who are occupied in mathematical education, were not aware of all of them before. Only when we met a dead end in proceeding with our constructions we turned to the textbooks and found in them unknown properties using which we were successful in dealing with the challenge of construction. We have no doubt that there are other properties allowing the constructions to be carried out using other ways. In any case it is worthy to ask the students to find additional properties in textbooks and articles on the subject, or to hypothesize and verify using dynamic geometrical software. However, the geometrical software is not sufficient, and any property must be proven using the formal method of mathematics.

4. Summary

Different methods were proposed to carry out geometrical constructions in the context of three loci that appear in the program of studies of high school education. Execution of the constructions relied upon their special properties, and in order to emphasize them applets were included which allows the reader to carry out the dynamic investigation thereof.

5. Acknowledgements

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